Classical Computers Very Likely Can *Not* Efficiently Simulate Multimode Linear Optical Interferometers with Arbitrary Fock–State Inputs—An Elementary Argument

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Aaronson and Arkhipov recently used computational complexity theory to argue that classical computers very likely cannot efficiently simulate linear, multimode, quantum–optical interferometers with arbitrary Fock–state inputs [S. Aaronson and A. Arkhipov, Theory of Computing, **9**, 143 (2013)]. Here we present an elementary argument that utilizes only techniques from quantum optics. We explicitly construct the Hilbert space for such an interferometer and show that that its dimension scales exponentially with all the physical resources. Finally we also show in a simple example just how the Schrödinger and Heisenberg pictures of quantum theory, while mathematically equivalent, are not in general computationally equivalent.

There is a history of attempts to use linear quantum interferometers to design a quantum computer. Černý showed that a linear interferometer could solve NP–complete problems in polynomial time but only with an exponential overhead in energy [1]. Clauser and Dowling showed that a linear interferometer could factor large numbers in polynomial time but only with exponential overhead in both energy and spatial dimension [2]. Cerf, Adami, and Kwiat showed how to build a programmable linear quantum optical computer but with an exponential overhead in spatial dimension [3].

Nonlinear optics provides a well–known route to universal quantum computing [4]. We include in this nonlinear class the so–called 'linear' optical approach to quantum computing [5], because this scheme contains an effective Kerr nonlinearity [6].

In light of these results there arose a widely held belief that linear interferometers alone, even with nonclassical input states, cannot provide a road to universal quantum computation and, as a corollary, that all such devices can be efficiently simulated classically. However, recently Aaronson and Arkhipov (AA) gave an argument that multimode, linear, quantum optical interferometers with arbitrary Fock–state photon inputs very likely could not be simulated efficiently with a classical computer [7]. Their argument, couched in the language of quantum computer complexity class theory, is not easy to follow for those not skilled in that art. Nevertheless, White, collaborators, and several other groups carried out experiments that demonstrated that the conclusion of AA holds up for small photon numbers [8–11]. Our goal here is to understand—from a physical point of view—why such a device cannot be simulated classically.

We independently came to the same conclusion as AA in our recent analysis of multi–photon quantum random walks in a particular multimode interferometer called a quantum "pachinko" machine shown in Fig.1 [12]. The dual–photon Fock state $|N\rangle\,|M\rangle$ is inputted into the top of the interferometer and then the photons are allowed to cascade downwards through the lattice of beam splitters (B) and phase shifters (φ) to arrive at an array of photon–number–resolving detectors (∇) at the bottom. Our goal (that of a pachinko player) was to calculate the probability that p photons were detected at the q^{th} detector for arbitrary input photon number and lattice dimension. We failed utterly. It is easy to see why.

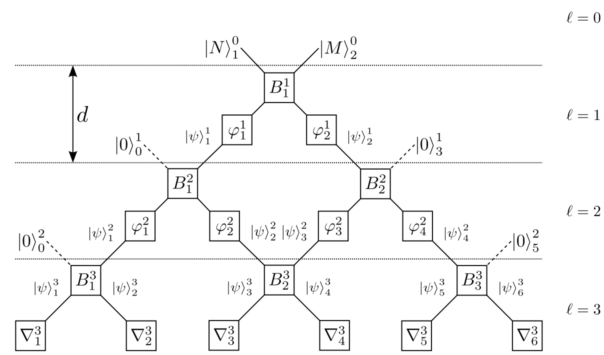

FIG. 1. The quantum pachinko machine for numerical depth $L = 3$. We indicate an arbitrary bosonic dual–Fock input $|N\rangle\,|M\rangle$ at the top of the interferometer and then the lattice of beam splitters (B), phase shifters (φ), and photon–number–resolving detectors (∇). The vacuum input modes $|0\rangle$ (dashed lines) and internal modes $|\psi\rangle$ (solid lines) are also shown. The notation is such that the superscripts label the level ℓ and the subscripts label the row element from left to right.

Working in the Schrödinger picture, we set out to compute the probability amplitudes at each detector by following the Feynman–like paths of each photon through the machine, and then summing their contributions at the end. For a machine of numerical depth L, as shown

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in Fig. 1, it is easy to compute that the number of such Feynman–like paths is $2^{L(N+M)}$. So for even a meager number of photons and levels the solution to the problem by this Schrödinger picture approach becomes rapidly intractable. For example, choosing $N = M = 9$ and $L = 6$, we have $2^{288} \cong 5 \times 10^{86}$ total possible paths, which is about four orders of magnitude larger then the number of atoms in the observable universe. We were puzzled by this conclusion; we expected any passive linear quantum optical interferometer to be efficiently simulatable classically. With the AA result now in hand, we set out here to investigate the issue of the complexity of our quantum pachinko machine from an intuitive physical perspective. The most mathematics and physics we shall need is elementary combinatorics and quantum optics.

Following Feynman, we shall explicitly construct the pachinko machine's Hilbert state space for an arbitrary level L, and for arbitrary photon input number, and show that the space's dimension grows exponentially as a function of each of the physical resources needed to build and run the interferometer [13]. From this result we conclude that it is very likely that any classical computer that tries to simulate the operation of the quantum pachinko machine will always suffer an exponential slowdown. We will also show that no exponential growth occurs if Fock states are replaced with photonic coherent states or squeezed states, which elucidates the special nature of photonic Fock states.

As our argument is all about counting resources, we have carefully labeled all the components in the pachinko machine in Fig. 1 to help us with that reckoning. The machine has a total of L levels of physical depth d each. The input state at the top is the dual–Fock state $|N\rangle_1^0 |M\rangle_2^0$, where the superscripts label the level number and the subscripts the element in the row at that level (from left to right). We illustrate a machine of total numerical depth of $L = 3$. For $1 \leq \ell < L$, we show the vacuum input modes along the edges of the machine. The resources we are most concerned about are energy, time, spatial dimension, and number of physical elements needed to construct the device. All of these scale either linearly or quadratically in either L or $N + M$. The total physical depth is $D = Ld$ and so the spatial area is $A = (\sqrt{2}D)^2 = 2L^2d^2$. Using identical photons of frequency ω, the energy per run is $E = (N + M)\hbar\omega$. The time it takes for the photons to arrive at the detectors is $T = \sqrt{2}Ld/c$, where c is the speed of light. In each level the photons encounter ℓ number of BS so the total number is $\#B = \sum_{\ell=1}^{L} \ell = L(L + 1)/2$. Below each BS (with the exception of the L^{th} level) there are two independently tunable PS for a total number of PS that is $\#\varphi = \sum_{\ell=1}^{L-1} 2\ell = L(L - 1)$. The total number of detectors is $\#\nabla = 2L$. The total number of input modes is equal to the total number of output modes and is $\#I = \#O = 2L$. The total number of internal modes is $\#\psi = \sum_{\ell=1}^{L-1} 2\ell = L(L - 1)$. As promised everything scales either linearly or quadratically in either L or $N + M$.

The input state may be written in the Heisenberg picture as $|N\rangle_1^0 |M\rangle_2^0 = (\hat{a}_1^{\dagger 0})^N (\hat{a}_2^{\dagger 0})^M |0\rangle_1^0 |0\rangle_2^0 / \sqrt{N!M!}$, where $\hat{a}^\dagger$ is a modal creation operator. Each BS performs a forward unitary mode transformation, which we illustrate with B_1^1, of the form $\hat{a}_1^1 = ir_1^1 \hat{a}_1^0 + t_1^1 \hat{a}_2^0$ and $\hat{a}_2^1 = t_1^1 \hat{a}_1^0 + ir_1^1 \hat{a}_2^0$ where the reflection and transmission coefficients r and t are positive real numbers such that $r^2 + t^2 = R + T = 1$. The choice $r = t = 1/\sqrt{2}$ implements a 50–50 BS. Each PS is implemented by, for example, applying the unitary operation $\exp(i\varphi_1^1 \hat{n}_1^1)$ on mode $|\psi\rangle_1^1$, where $\hat{n}_1^1 := \hat{a}_1^{\dagger 1} \hat{a}_1^1$ is the number operator, $\hat{a}_1^1$ is the annihilation operator conjugate to $\hat{a}_1^{\dagger 1}$, and φ_1^1 is a real number. Finally the $2L$ detectors in the final level L are each photon number resolving [14].

To argue that this machine (or any like it) cannot be simulated classically, in general, it suffices to show that this is so for a particular simplified example. We now take N and L arbitrary but $M = 0$ and turn off all the phase shifts and make all the BS identical by setting $\varphi_k^\ell = 0$, $t_k^\ell = t$, and $r_k^\ell = r$ for all (k, ℓ). We then need the backwards BS transformation on the creation operators, which is, $\hat{a}_1^{\dagger 0} = ir\hat{a}_1^{\dagger 1} + t\hat{a}_2^{\dagger 1}$ and $\hat{a}_2^{\dagger 0} = t\hat{a}_1^{\dagger 1} + ir\hat{a}_2^{\dagger 1}$. Similar transforms apply down the machine at each level. With $M = 0$ the input simplifies to $|N\rangle_1^0 |0\rangle_2^0 = (\hat{a}_1^{\dagger 0})^N |0\rangle_1^0 |0\rangle_2^0 / \sqrt{N!}$ and now we apply the first backwards BS transformation $|\psi\rangle_1^1 |\psi\rangle_2^1 = (ir\hat{a}_1^{\dagger 1} + t\hat{a}_2^{\dagger 1})^N |0\rangle_1^0 |0\rangle_2^0 / \sqrt{N!}$ to get the state at level one.

At every new level each $\hat{a}^\dagger$ will again bifurcate according to the BS transformations for that level, with the total number of bifurcations equal to the total number of BS, and so the computation of all the terms at the final level involves a polynomial number of steps in L. It is instructive to carry this process out explicitly to level $L = 3$ to get,

$$
\begin{aligned}
|\psi\rangle^3 = \frac{1}{\sqrt{N!}} (irt^2 \hat{a}_1^{\dagger 3} - r^2 t \hat{a}_2^{\dagger 3} + ir(t^2 - r^2) \hat{a}_3^{\dagger 3} \\
- 2r^2 t \hat{a}_4^{\dagger 3} + irt^2 \hat{a}_5^{\dagger 3} + t^3 \hat{a}_6^{\dagger 3})^N \prod_{\ell=1}^{6} |0\rangle_\ell^3 ,
\end{aligned}
\tag{1}
$$

where we have used a tensor product notation for the states. If $r \cong 0$ or $r \cong 1$ the state is easily computed. Since we are seeking a regime that cannot be simulated classically we work with $r \cong t \cong 1/\sqrt{2}$.

It is now clear from Eq.(1) what the general form of the solution will be. We define

$$
|\psi\rangle^L := \sum_{\substack{\{n_\ell\} \\ N = \sum_{\ell=1}^{2L} n_\ell}} |\psi\rangle_{\{n_\ell\}}^L \quad ; \quad |0\rangle^L := \prod_{\ell=1}^{2L} |0\rangle_\ell^L \tag{2}
$$

and the general solution has the form,

$$
\begin{aligned}
|\psi\rangle^L &= \frac{1}{\sqrt{N!}}\left(\sum_{\ell=1}^{2L}\alpha_\ell^L \hat{a}_\ell^{\dagger L}\right)^N |0\rangle^L \\
&= \frac{1}{\sqrt{N!}}\sum_{N=\sum_{\ell=1}^{2L} n_\ell}\binom{N}{n_1, n_2, \ldots, n_{2L}} \\
&\times \prod_{1\le k\le 2L}(\alpha_k^L \hat{a}_k^{\dagger L})^{n_k}|0\rangle^L ,
\end{aligned}
\tag{3}
$$

where all the coefficients α_ℓ^L will be nonzero in general. Since all the operators commute, as they each operate on a different mode, we have expanded Eq.(3) using the multinomial theorem where the sum in the expansion is over all combinations of non–negative integers constrained by $N = \sum_{\ell=1}^{2L} n_l$ and

$$
\binom{N}{n_1, n_2, \ldots, n_{2L}} = \frac{N!}{n_1! n_2! \ldots n_{2L}!}, \tag{4}
$$

is the multinomial coefficient [15]. The state $|\psi\rangle^L$ is highly entangled over the number–path degrees of freedom. Each monomial in the expansion of Eq.(3) is unique and so the action of the set of all monomial operators on the vacuum will produce a complete orthonormal basis set for the Hilbert space at level L, given by $|\psi\rangle^L_{\{n_\ell\}} := \prod_{\ell=1}^{2L}|n_\ell\rangle^L_\ell$, where the n_ℓ are subject to the same sum constraint. Let us call the dimension of that Hilbert space $\dim[H(N,L)]$, which is therefore the total number of such basis vectors.

Taking $L = 3$ and $N = 2$, we can use Eq.(3) to compute the probability a particular sequence of detectors will fire with particular photon numbers. What is the probability detector one gets one photon, detector two also gets one, and all the rest get zero? This is the modulus squared of the probability amplitude of the state $|1\rangle_1^3 |1\rangle_2^3 |0\rangle_3^3 |0\rangle_4^3 |0\rangle_5^3 |0\rangle_6^3$. Setting $r = t = 1/\sqrt{2}$ for the 50–50 BS case, from Eq.(1) we read off $\alpha_1^3 = irt^2 = i/(2\sqrt{2})$ and $\alpha_2^3 = -r^2 t = -1/(2\sqrt{2})$, and so the probability of this event is given by $P_{110000} \cong 0.031$.

It turns out that it is possible (for general L and N) to compute the binary joint probabilities, that detector p gets n photons and detector q gets m [16]. However the pachinko player instead wants to know which detector gets *the most* photons on average. For $L = 3$ and $N = 2$, to compute this he must calculate all the $\dim[H(2,3)] = 21$ probabilities P^3_{npqrst} and then choose from that list which detector's likely to detect the most photons. However for arbitrary L and N this optimization he can not do, as we now show. We can provide a closed form expression for $\dim[H(N,L)]$ by realizing that it is the same as the number of different ways one can add up non–negative integers that total to fixed N. More physically this is the number of possible ways that N indistinguishable photons may be distributed over $2L$ detectors. The answer is well known in the theory of combinatorics and is:

$$
\dim[H(N,L)] = \binom{N+2L-1}{N}, \tag{5}
$$

where this is the ordinary binomial coefficient [17]. For our example with $L = 3$, $N = 2$, Eq.(5) implies that the number of distinct probabilities P^3_{npqrst} to be tabulated is again 21.

We first examine two "computationally simple" examples. Taking N arbitrary and $L = 1$ we get $\dim[H(N,1)] = N + 1$, which is easily seen to be the number of ways to distribute N photons over two detectors. Next taking $N = 1$ and L arbitrary we get $\dim[H(1,L)] = 2L$, which is the number of ways to distribute a single photon over $2L$ detectors. If we were to invoke Dirac's edict—"Each photon then interferes only with itself."—we would then expect that adding a second photon should only double this latter result [18]. Instead the effect of two–photon interference on the state space can be seen immediately by computing $\dim[H(2,L)] = L(2L+1)$. That is adding a second photon causes a quadratic (as opposed to linear) jump in the size of the Hilbert space. Dirac was wrong, photons do interfere with each other, and that multiphoton interference directly affects the computational complexity. All these three cases are simulatable in polynomial time steps with N and L, but we see a quadratic jump in dimension as soon as we go from one to two photons. These jumps in complexity continue for each additional photon added and the dimension grows rapidly.

We therefore next investigate a "computationally complex" intermediate regime by fixing $N = 2L - 1$. That is we build a machine with total number of levels L and then choose an odd–numbered photon input so that this restriction holds. Eq.(5) becomes $\dim[H(N)] = (2N)!/(N!)^2$. Deploying Sterling's approximation for large N, in the form $n! \cong (n/e)^n\sqrt{2\pi n}$ we have $\dim[H(N)] \cong 2^{2N}/\sqrt{\pi N}$. This is our primary result. The Hilbert space dimension scales exponentially with $N = 2L - 1$. Since all the physical parameters needed to construct and run our quantum pachinko machine scale only linear or quadratically with respect to N or L, we have an exponentially large Hilbert space produced from a polynomial number of physical resources —Feynman's necessary condition for a potential universal quantum computer.

Let us suppose we build onto an integrated optical circuit a machine of depth $L = 69$ and fix $N = 2L-1 = 137$. Such a device is not too far off on the current quantum optical technological growth curve [19]. Then we have $\dim[H(137)] = 10^{81}$, which is again on the order of the number of atoms in the observable universe. Following Feynman's lead, we conclude that, due to this exponentially large Hilbert space, it is likely that a classical computer can *not* efficiently simulate our quantum pachinko machine, in particular, and hence it likely can not in general efficiently simulate all multimode linear optical interferometers with arbitrary Fock–state inputs.

Let us now compare our Heisenberg picture result to that of the Schrödinger picture. In the computationally complex regime where $N = 2L-1$ the number of distinct Feynman–like paths we must follow in the Schrödinger picture is $2^{LN} = 2^{N(N+1)/2} \cong 2^{N^2/2}$. Taking $N = 137$ and $L = 69$, as in the previous example, we get an astounding $2^{9453} \cong 4 \times 10^{2845}$ total paths. Dirac proved that the Heisenberg and Schrödinger pictures are mathematically equivalent, that they always give the same predictions, but we see here that they are not always necessarily *computationally* equivalent [20]. Calculations in the Heisenberg picture are often *much* simpler than in the Schrödinger picture.The fact that the two pictures are not computationally equivalent is implicit in the Gottesman-Knill theorem, however it is satisfying to see here just how that is so in a simple optical interferometer [21].

To contrast this exponential overhead from the resource of bosonic Fock states, let us now carry out the same analysis with the bosonic coherent input state input $|\beta\rangle_1^0 |0\rangle_2^0$, where we take the mean number of photons to be $|\beta|^2 = \overline{n}$. In the Heisenberg picture this input becomes $\hat{D}_1^0(\beta) |0\rangle_1^0 |0\rangle_2^0$ where $\hat{D}_1^0(\beta) = \exp(\beta \hat{a}_1^{\dagger 0} - \beta^* \hat{a}_1^0)$ is the displacement operator [22]. Applying the BS transformations down to final level L we get,

$$
\begin{aligned}
|\psi\rangle^L &= \exp\left(\beta \sum_{\ell=1}^{2L} \alpha_\ell^L \hat{a}_\ell^{\dagger L} - \beta^* \sum_{\ell=1}^{2L} \alpha_\ell^{L*} \hat{a}_\ell^L\right) |0\rangle^L \\
&= \prod_{\ell=1}^{2L} \exp(\beta \alpha_\ell^L \hat{a}_\ell^{\dagger L} - \beta^* \alpha_\ell^{L*} \hat{a}_\ell^L) |0\rangle^L \qquad (6) \\
&= \prod_{\ell=1}^{2L} |\beta \alpha_\ell^L\rangle_\ell^L ,
\end{aligned}
$$

At the output we have $2L$ coherent states that have been modified in phase and amplitude. This is to be expected, as it is well known that linear interferometers transform a coherent state into another coherent state. Since all the coefficients α_ℓ^L are computable in $\#B = L(L+1)/2$ steps, this result is obtained in polynomial time steps in L, independent of $\overline{n}$. The mean number of photons at each detector is then simply $\overline{n}_\ell^L = |\beta \alpha_\ell^L|^2 = \overline{n} |\alpha_\ell^L|^2$.

A similar analysis may be carried out for bosonic squeezed input states. Taking, for example, a single–mode squeezed vacuum input $|\xi\rangle_1^0 |0\rangle_2^0 = \hat{S}_1^0(\xi) |0\rangle_1^0 |0\rangle_2^0$, with the squeezing operator defined as $\hat{S}_1^0(\xi) = \exp[(\xi^* (\hat{a}_1^0)^2 - \xi (\hat{a}_1^{\dagger 0})^2)/2]$, we arrive at,

$$
|\psi\rangle^L = \exp[(\xi^* (\sum_{\ell=1}^{2L} \alpha_\ell^* \hat{a}_\ell^L)^2 - \xi (\sum_{\ell=1}^{2L} \alpha_\ell \hat{a}_\ell^{\dagger L})^2)/2] |0\rangle^L , \qquad (7)
$$

which does not in general decompose into a separable product of single–mode squeezers on each output port. Nevertheless the probability amplitudes may still be computed in a time polynomial in L by noting that, from Eq.(5) with $N = 2$, there are at most $2L(L+1)$ terms in this exponent that must be evaluated. This result generalizes to arbitrary squeezed–state inputs [23]. The output of the interferometer may be then calculated on the transformed device in polynomial steps in L.

The exponential scaling comes from the bosonic Fock structure $|N\rangle = (\hat{a}^\dagger)^N |0\rangle / \sqrt{N!}$ and the fact that N can be made close to L with identical bosons in a single mode. It is well known that beam splitters can generate number–path entanglement from separable bosonic Fock states. For example, the simplest version of the Hong–Ou–Mandel effect at level one with separable input $|1\rangle_1^0 |1\rangle_2^0$ becomes $|\psi\rangle_1^1 |\psi\rangle_2^1 = (i\hat{a}_1^{\dagger 1} + \hat{a}_2^{\dagger 1})(\hat{a}_1^{\dagger 1} + i\hat{a}_2^{\dagger 1}) |0\rangle_1^1 |0\rangle_2^1 /2 = i[|2\rangle_1^1 |0\rangle_2^1 + |0\rangle_1^1 |2\rangle_2^1]/\sqrt{2}$ a N00N state [24]. Such entangled N00N states violate a Bell inequality and are hence nonlocal even though the input was not [25]. For arbitrary bosonic Fock input states and interferometer size the amount of number–path entanglement grows exponentially fast.

In conclusion, we have shown that a multimode linear optical interferometer with arbitrary Fock input states is very likely not simulatable classically. Our result is consistent with the argument of AA. Without invoking complexity theory, we have argued this by explicitly constructing the Hilbert state space of a particular such interferometer and showed that the dimension grows exponentially with the size of the machine. The output state is highly entangled in the photon number and path degrees of freedom. We have also shown that simulating the device has radically different computational overheads in the Heisenberg versus the Schrödinger picture, illustrating just how the the two pictures are not in general computationally equivalent within this simple linear optical example. It is unknown if such bosonic multi–mode interferometers as these are universal quantum computers, but regardless they will certainly not be fault tolerant. As pointed out by Rohde [26], it is well known that Fock states of high photon number are particularly sensitive to loss [27]. They are also super–sensitive to dephasing as well [28]. This implies that even if such a machine turns out to be universal it would require some type of error correction to run fault tolerantly. Nevertheless such devices could be interesting tools for studying the relationship between multiphoton interference and quantum information processing for small numbers of photons. If we choose each of the PS and BS transformations independently of each other, we have a mechanism to program the pachinko machine by steering the output into any of the possible output states. Even if universality turns out to be lacking we may very well be able to exploit this programmability to make a special purpose quantum simulator for certain physics problems such as frustrated spin systems [29].

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